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Nonstandard coproducts and the Izergin-Korepin open spin chain

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Abstract

Corresponding to the Izergin-Korepin $(A_2^{(2)})$ R matrix, there are three diagonal solutions ("K matrices") of the boundary Yang-Baxter equation. Using these R and K matrices, one can construct transfer matrices for open integrable quantum spin chains. The transfer matrix corresponding to the identity matrix $K = \mathbb{I}$ is known to have $U_q(o(3))$ symmetry. We argue here that the transfer matrices corresponding to the other two K matrices also have $U_q(o(3))$ symmetry, but with a nonstandard coproduct. We briefly explore some of the consequences of this symmetry.

1 Introduction and summary

The notion of coproduct is of fundamental importance in the theory of representations of algebras. Given a representation of an algebra on a vector space V, the coproduct Δ determines the representation on the tensor product space $V \otimes V$. For a classical Lie algebra, the coproduct is trivial: for any generator x, the coproduct is $\Delta(x) = x \otimes \mathbb{I} + \mathbb{I} \otimes x$, where $\mathbb{I}$ is the identity matrix. For quantum algebras, the situation is more interesting. Indeed, consider the case $U_q(su(2))$, with a set of three generators $\{j_\pm \,, h\}$ obeying

$$[h \,, j_\pm] = \pm j_\pm \,. \tag{1}$$

As is well known, the "standard" coproduct

$$\begin{aligned} \Delta(h) &= h \otimes \mathbb{I} + \mathbb{I} \otimes h \,, \\ \Delta(j_\pm) &= j_\pm \otimes q^h + q^{-h} \otimes j_\pm \,, \end{aligned} \tag{2}$$

is compatible with the commutation relation

$$[j_+ \,, j_-] = \frac{q^{2h} - q^{-2h}}{q - q^{-1}} \,. \tag{3}$$

Perhaps less well-known is the fact that there is also a "nonstandard" coproduct

$$\begin{aligned} \Delta(h) &= h \otimes \mathbb{I} + \mathbb{I} \otimes h \,, \\ \Delta(j_\pm) &= j_\pm \otimes \mathbb{I} + q^h \otimes j_\pm \,, \end{aligned} \tag{4}$$

which is compatible instead with the q-commutation relation

$$j_+ j_- - q^{-1} j_- j_+ = \frac{\mathbb{I} - q^{2h}}{1 - q^2} \,. \tag{5}$$

Remarkably, both of these types of coproducts can be realized in the open integrable quantum spin chain constructed with the $A_2^{(2)}$ R matrix [1] by choosing appropriate boundary conditions. Let us briefly recall the history of this model. Sklyanin [2] pioneered the generalization of the Quantum Inverse Scattering Method [3] to systems with boundaries, and showed that integrable boundary conditions can be obtained from solutions $K(u)$ of the boundary Yang-Baxter equation [4], [5]. This approach was then generalized [6] to spin chains associated with general affine Lie algebras [7], [8]. In particular, for the $A_2^{(2)}$ case, it was found [9] that there are only three diagonal solutions of the boundary Yang-Baxter equation:

$$K^{(0)}(u) = \mathbb{I} = diag\Big(1 \,, 1 \,, 1\Big) \,,$$

$$
\begin{aligned}
K^{(1)}(u) &= diag\Big(e^{-u}\,, \frac{\sinh(\frac{1}{2}(3\eta - \frac{i\pi}{2} + u))}{\sinh(\frac{1}{2}(3\eta - \frac{i\pi}{2} - u))}\,, e^{u}\Big)\,, \\
K^{(2)}(u) &= diag\Big(e^{-u}\,, \frac{\cosh(\frac{1}{2}(3\eta - \frac{i\pi}{2} + u))}{\cosh(\frac{1}{2}(3\eta - \frac{i\pi}{2} - u))}\,, e^{u}\Big)\,,
\end{aligned} \tag{6}
$$

where u is the spectral parameter, and η is the anisotropy parameter. Let us denote the corresponding transfer matrices for open quantum spin chains with N sites by $t^{(i)}(u)$, $i = 0\,,1\,,2$. (The construction of these transfer matrices is described below in Section 2.) It was shown in [10], [11] that the transfer matrix $t^{(0)}(u)$ constructed with the identity matrix $K^{(0)}$ has $U_q(o(3))$ symmetry:

$$
\big[t^{(0)}(u)\,, S^{\pm}\big] = 0\,, \qquad \big[t^{(0)}(u)\,, S^{3}\big] = 0\,, \tag{7}
$$

where the generators obey

$$
\big[S^3\,, S^{\pm}\big] = \pm S^{\pm}\,, \qquad \big[S^+\,, S^-\big] = \frac{q^{2S^3} - q^{-2S^3}}{q - q^{-1}}\,, \tag{8}
$$

and

$$
S^{\pm} = \sum_{k=1}^{N} q^{s^3_N + \cdots + s^3_{k+1}}\ s^{\pm}_k\ q^{-(s^3_{k-1} + \cdots + s^3_1)}\,, \qquad S^3 = \sum_{k=1}^{N} s^3_k\,, \tag{9}
$$

where

$$
s^+ = \sqrt{2\cosh\eta}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad s^- = \sqrt{2\cosh\eta}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad s^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}\,, \tag{10}
$$

and $q = e^{\eta}$. That is, the transfer matrix has quantum algebra symmetry with the "standard" coproduct (2). This is a generalization of the observation [12], [13] of $U_q(su(2))$ symmetry for the $A_1^{(1)}$ case. Batchelor and Yung [14] later showed that the open $A_2^{(2)}$ spin chain can be mapped to the problem of polymers at surfaces, and that the above three solutions $K^{(i)}(u)$ correspond to three distinct surface critical behaviors.

There has remained the question: what symmetry – if any – do the transfer matrices constructed with $K^{(1)}$ and $K^{(2)}$ have? Naively, one expects that since $K \neq \mathbb{I}$, there is less symmetry. [1] However, this is *not* the case. We argue here that the transfer matrices $t^{(1)}(u)$ and $t^{(2)}(u)$ also have $U_q(o(3))$ symmetry, but with a "nonstandard" coproduct (4):

$$
\big[t^{(i)}(u)\,, S^{\pm}\big] = 0\,, \qquad \big[t^{(i)}(u)\,, S^{3}\big] = 0\,, \qquad i = 1\,,2\,, \tag{11}
$$

[1] This expectation holds true for the $A_n^{(1)}$ case [15]. Indeed, there the diagonal K matrices contain an additional continuous parameter ξ; and $K = \mathbb{I}$ is a point ($\xi \to \infty$) of enhanced symmetry.

where the generators obey

$$\left[S^3\,, S^{\pm}\right] = \pm 2 S^{\pm}\,, \qquad S^+ S^- - q^{-2} S^- S^+ = \frac{\mathbb{I} - q^{2S^3}}{1-q^2}\,, \tag{12}$$

and

$$S^{\pm} = \sum_{k=1}^{N} s_k^{\pm}\; q^{s_{k-1}^3 + \cdots + s_1^3}\,, \qquad S^3 = \sum_{k=1}^{N} s_k^3\,, \tag{13}$$

where

$$s^+ = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)\,, \quad s^- = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right)\,, \quad s^3 = \left(\begin{array}{ccc} 1 & & \\ & 0 & \\ & & -1 \end{array}\right)\,, \tag{14}$$

and $q = e^{4\eta}$. Knowledge of such symmetry is essential for understanding important features of the models such as degeneracies of the spectrum and the Bethe Ansatz solution. Eqs. (11) - (14) are the main results of this Letter. In Section 2 we provide some pertinent details about the construction and symmetry of the models, and we conclude in Section 3 with a brief discussion.

2 Some details

In this Section, we briefly review the construction of the transfer matrices, and outline the argument for their symmetry. The solution $R(u)$ of the Yang-Baxter equation found by Izergin and Korepin [1], which corresponds [7],[8] to the case $A_2^{(2)}$, can be written in the following form [16], [11]

$$R(u) = \left(\begin{array}{ccc|ccc|ccc}
c & & & & & & & & \\
 & b & & e & & & & & \\
 & & d & & g & & f & & \\
\hline
 & \bar{e} & & b & & & & & \\
 & & \bar{g} & & a & & g & & \\
 & & & & & b & & e & \\
\hline
 & & \bar{f} & & \bar{g} & & d & & \\
 & & & & & \bar{e} & & b & \\
 & & & & & & & & c
\end{array}\right) \tag{15}$$

where

$$a = \sinh(u-3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh \eta\,, \qquad b = \sinh(u-3\eta) + \sinh 3\eta\,,$$

$$
\begin{aligned}
c &= \sinh(u-5\eta)+\sinh\eta\,, & d &= \sinh(u-\eta)+\sinh\eta\,,\\
e &= -2e^{-\frac{u}{2}}\sinh 2\eta\ \cosh(\frac{u}{2}-3\eta)\,, & \bar{e} &= -2e^{\frac{u}{2}}\sinh 2\eta\ \cosh(\frac{u}{2}-3\eta)\,,\\
f &= -2e^{-u+2\eta}\sinh\eta\ \sinh 2\eta - e^{-\eta}\sinh 4\eta\,, & \bar{f} &= 2e^{u-2\eta}\sinh\eta\ \sinh 2\eta - e^{\eta}\sinh 4\eta\,,\\
g &= 2e^{-\frac{u}{2}+2\eta}\sinh\frac{u}{2}\ \sinh 2\eta\,, & \bar{g} &= -2e^{\frac{u}{2}-2\eta}\sinh\frac{u}{2}\ \sinh 2\eta\,.
\end{aligned}
$$

It has the regularity property $R(0) \propto \mathcal{P}$, where $\mathcal{P}$ is the permutation matrix, as well as unitarity, PT symmetry, and crossing symmetry

$$
R_{12}(u) = V_1\ R_{12}(-u-\rho)^{t_2}\ V_1 = V_2^{t_2}\ R_{12}(-u-\rho)^{t_1}\ V_2^{t_2}\,, \tag{16}
$$

where the crossing matrix V is given by

$$
V = \begin{pmatrix} & & -e^{-\eta} \\ & 1 & \\ -e^{\eta} & & \end{pmatrix}\,, \tag{17}
$$

and $\rho = -6\eta - i\pi$.

Given a solution $K(u)$ of the boundary Yang-Baxter equation, a corresponding transfer matrix $t(u)$ for an open integrable quantum spin chain with N sites is given by [2], [6], [15]

$$
t(u) = \mathrm{tr}_0\, M_0 K_0(-u-\rho)^{t_0} \mathcal{T}_0(u)\,, \tag{18}
$$

where

$$
\mathcal{T}_0(u) = T_0(u)\ K_0(u)\ \hat{T}_0(u)\,, \tag{19}
$$

with

$$
T_0(u) = R_{0N}(u)\cdots R_{01}(u)\,, \qquad \hat{T}_0(u) = R_{10}(u)\cdots R_{N0}(u)\,, \tag{20}
$$

and

$$
M = V^t V = diag\Big(e^{2\eta}\,, 1\,, e^{-2\eta}\Big)\,. \tag{21}
$$

Indeed, the transfer matrix forms a one-parameter commutative family $[t(u)\,, t(v)] = 0$, which contains the Hamiltonian $\mathcal{H}$,

$$
\mathcal{H} \propto \frac{d}{du} t(u)\Big|_{u=0}\,. \tag{22}
$$

For the three K matrices $K^{(i)}(u)$ given in Eq. (6), we denote by $t^{(i)}(u)$ the corresponding transfer matrices, and by $\mathcal{H}^{(i)}$ the corresponding Hamiltonians. We now restrict our attention to the cases $i = 1\,, 2$. For 2 sites ($N = 2$), we have checked the $U_q(o(3))$ symmetry (11) - (14) of the transfer matrix by direct computation. In particular, Eq. (22) implies that the 2-site Hamiltonian also has this symmetry. For general N, the Hamiltonian is given by a sum of 2-site Hamiltonians plus boundary terms. It follows that, for general N, the Hamiltonian $\mathcal{H}^{(i)}$ has $U_q(o(3))$ symmetry

$$\left[\mathcal{H}^{(i)}\,, S^{\pm} \right] = 0\,, \qquad \left[\mathcal{H}^{(i)}\,, S^{3} \right] = 0\,, \qquad i = 1\,, 2\,, \tag{23}$$

where the symmetry generators obey (12) - (14). We have also checked the symmetry (11) of the transfer matrix for 3 sites ($N = 3$) by direct computation, and we conjecture that it holds for general N.

We remark that the symmetry generators $S^{\pm}$, S^3 defined in (13), (14) lie in the fundamental algebraic structures of QISM. Indeed, note the asymptotic behavior of the R and K matrices for $u \rightarrow \infty$:

$$R(u) \;\sim\; e^u R^+ + R^{++} + O(e^{-u})\,, \tag{24}$$

$$K^{(i)}(u) \;\sim\; e^u K^{(i)+} + K^{(i)++} + O(e^{-u})\,, \qquad i = 1\,, 2\,, \tag{25}$$

where $R^+\,, R^{++}\,, K^{(i)+}\,, K^{(i)++}$ are independent of u. It follows that the quantity $\mathcal{T}^{(i)}(u)$ defined as in Eq. (19) has the asymptotic behavior for $u \rightarrow \infty$

$$\mathcal{T}^{(i)}(u) \sim e^{(2N+1)u} \mathcal{T}^{(i)+} + e^{2Nu} \mathcal{T}^{(i)++} + \dots\,, \tag{26}$$

where $\mathcal{T}^{(i)+}\,, \mathcal{T}^{(i)++}$ are independent of u. The basic observation is that the generators $S^{\pm}$ lie in the antidiagonal corners of $\mathcal{T}^{(i)++}$ (viewed as a 3×3 auxiliary-space matrix, with operator-valued entries):

$$\mathcal{T}^{(i)++} = \left(\begin{array}{ccc} 0 & 0 & S^- \\ 0 & * & * \\ S^+ & * & * \end{array} \right) . \tag{27}$$

We expect that this observation will be useful for formulating a QISM proof of the symmetry (11).

3 Discussion

One immediate consequence of the symmetry which we have uncovered is the explanation of degeneracies in the spectrum for finite N. For instance, consider the pseudovacuum vector

$$\omega = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^{\otimes N} \,,$$

$$t^{(i)}(u)\ \omega = \Lambda^{(i)}(u)\ \omega \,, \qquad i = 1\,, 2\,, \tag{28}$$

where $\Lambda^{(i)}(u)$ is the corresponding pseudovacuum eigenvalue. Commutativity of the transfer matrix with S^- implies that the vectors $(S^-)^n \omega$ for $n = 1\,, 2\,, \ldots\,, N$ are also eigenvectors of the transfer matrix with the same eigenvalue.

Note that the pseudovacuum vector ω is annihilated by S^+; that is, $S^+ \omega = 0$. We expect that all Bethe Ansatz states (which can presumably be constructed by applying appropriate creation-like operators to ω) are such highest-weight states. (See, e.g., [17], [10], [18], [19].)

Finally, we remark that we have considered here only the first of the infinite family of models $A^{(2)}_{2n}$, $n = 1\,, 2\,, \ldots$. For these R matrices [7], [8], there are again only three distinct diagonal solutions of the boundary Yang-Baxter equation: $K^{(0)} = \mathbb{I}$ [9], and $K^{(1)}\,, K^{(2)}$ given in [20]. The transfer matrix constructed with $K^{(0)}$ has [10] the symmetry $U_q(o(2n+1))$ with the standard coproduct. We expect that the transfer matrices constructed with $K^{(1)}$ and $K^{(2)}$ also have $U_q(o(2n+1))$ symmetry, but with a nonstandard coproduct. We hope to report on this and related matters in a future publication.

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